Astronomy and Astrophysics
-Theoretical Guide-

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Bucharest, 2023

Since we astronomers are priests of the highest God in regard to the book of nature, it benefits us to be thoughtful, not of the glory of our minds, but rather, above all else, of the glory of God.

Johannes Kepler

Preface

Over the years, people have become more and more accustomed to the field of Astronomy. But not too long ago, it was a subject approached only by the nobility. But how did Astronomy become so mainstream? Let us first take a step back and ask what Astronomy even is and how it came to be what we know it is today.

Not to be confused nor united with the practices of Astrology (as it was until the 18th century), Astronomy is the oldest of all natural sciences, with roots in antiquity. In the earliest cultures, people identified celestial bodies as the Moon and the Sun with deities and related these objects and their movements to certain phenomena.

The Ancient Greeks were the first to develop astronomy as a branch of mathematics and treat it with rigour. One of the most important astronomers of the time was a man named Ptolemy. He catalogued a total of forty-eight constellations (known today as the *Ptolemaic constellations*), relating the figures he identified with mythological creatures and deities. In addition to that, he was the first to develop a model to predict the movement of the stars, the so-called *Ptolemaic system*. The Indians, Middle-Eastern, Mesoamericans and Chinese also independently made many observations and wrote down what they saw, with significant contributions to the modern catalogue of stars and constellations.

Throughout the Middle Ages, the subject was only pursued by select individuals and taught alongside Astrology in the earliest universities. In 1543, a polish astronomer named Nicolaus Copernicus published an article entitled *De revolutionibus corpum coelestium*, in which he proposed a new theoretical model, the so-called heliocentric model. It was the first out of many revolutionary ideas. The further discoveries and research of Galileo Galilei, Tycho Brahe and Isaac Newton brought forth upon the world a new field of study: *Astrophysics*.

Together with the development of Mathematics and Science, Astronomy eventually broke apart from the practices of Astrology in the 18th century, being treated as an area of Physics. Throughout the 19th century, people have become aware that the picture of the Universe they knew had flaws that could not be explained. That was until a man named Albert Einstein came in 1905 and built upon Maxwell's Ideas. He came up with the *Theory of Special Relativity*, which poked holes in Newton's theory. Its only flaw was the lack of an explanation for Gravity. However, Einstein would publish an article on the *Theory of General Relativity*, which completely revolutionized our perception of the Universe.

Further observations by the likes of Hubble and the Space Race brought mankind closer to space than ever before. New research in computers and precision engineering got us to where we are now: searching, theorising, testing and reaching out to the Universe. How much have we achieved? Only 5% of the entire Cosmos is known to us. The rest is out there, waiting to be discovered.

Author's Message to the Readers

So what is the purpose of this book? It is nothing but a guide. A quick guide to where we are now and a learning tool for those who wish to find out more about what happens beyond our reach and who wish to pursue this field of study. For those of you who are still students, it aims to prepare you for participating in the *Olympiad of Astronomy and Astrophysics*, providing the necessary mathematical and scientific apparatus to make astronomers and astrophysicists out of you. I can only thank you for considering my effort was worth your time and congratulate you all for having this wonderful passion.

However, this is NOT a beginner's guide. In order to fully grasp the contents, one ought to master high school Mathematics and Physics. For further questions, I am open to conversations via e-mail, under the following address:

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Acknowledgements

I am very grateful for the unconditioned help and support of my beloved girlfriend, both of our families and my friends and advisors at the University of Bucharest's Faculty of Physics in Măgurele, Romania during the writing of this book. I am also very thankful to my Astronomy teacher in high school, who inspired me to go further with my passion for exploring the Universe.

June 2023 Mihai Dragomir

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Chapter 1

Mathematical Interlude

1.1 Spherical Trigonometry

Definition 1. Let $OXYZ$ be a scalene trihedron. If O is the center of a sphere $\mathcal{S}$, the trihedron's intersections with the surface of the sphere are three arches which meet two at each vertex, thus creating a *spherical triangle* on the surface of $\mathcal{S}$ (Fig. 1.1).

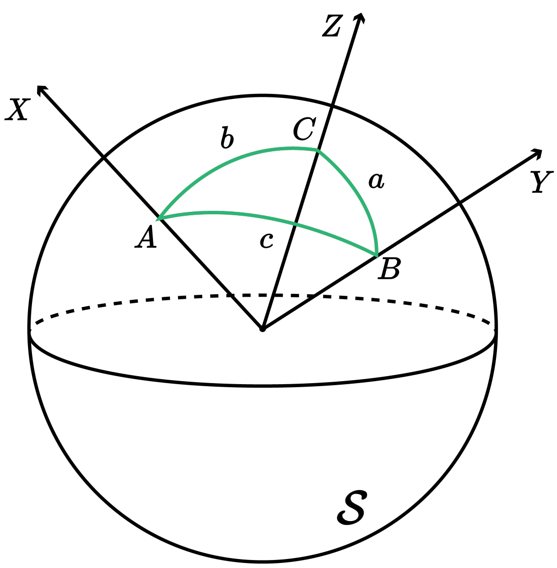

Figure 1.1: Image of the spherical triangle ΔABC

As a convention, we shall note the angles with uppercase letters and the arches with lowercase letters.

Proposition. For a spherical triangle ΔABC, the following inequalities are true:

$$\begin{cases} \pi < A + B + C < 3\pi \\ a + b + c < 2\pi \end{cases} \tag{1.1}$$

Proposition. For a spherical triangle ΔABC, we call the *semi-perimeter* and *excess* (noted as p and ε, respectively), the magnitudes satisfying the equations

$$\begin{cases} 2p = a + b + c \\ 2\varepsilon = A + B + C - \pi \end{cases} \tag{1.2}$$

Theorem. Spherical Laws of Sines and Cosines. Let ΔABC be a spherical triangle on a sphere $\mathcal{S}$. The following properties are satisfied:

i) $\cos a = \cos b \cos c + \sin b \sin c \cos A$ - Law of Cosines for Arches

ii) $\cos A = -\cos B \cos C + \sin B \sin C \cos a$ - Law of Cosines for Angles

iii) $\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$ - Law of Sines

Proof. i)

The trick used is expressing a dot product of two unit vectors. Considering that the properties of a spherical triangle are invariant to any scaling or rotation, we will rearrange the triangle in the following manner (Fig. 1.2):

- the sphere will have a unit radius;
- A will be in the North Pole of the sphere;
- the arch AB will be in the plane XOZ of a three-dimensional Cartesian frame centered in the center of the sphere.

It is useful now to establish a convention regarding vector notation. Throughout this book, usual vectors will be noted using bold letters (e.g. **F** for a force), except where it is impossible (e.g. $\vec{\omega}$, x, $\overrightarrow{\mathrm{OB}}$). Unit vectors will be noted with a hat (e.g. $\hat{\imath}$). We shall now consider the two unit vectors $\overrightarrow{\mathrm{OB}}$ and $\overrightarrow{\mathrm{OC}}$. Their coordinates in the frame $OXYZ$ are:

$$\begin{cases} \overrightarrow{\mathrm{OB}} = (\sin c, 0, \cos c) \\ \overrightarrow{\mathrm{OC}} = (\cos A \sin b, \sin A \sin b, \cos b) \end{cases}$$

$$\Rightarrow \overrightarrow{\mathrm{OB}} \cdot \overrightarrow{\mathrm{OC}} = \sin c \sin b \cos A + \cos c \cos b$$

On the other hand, the dot product of the two unit vectors can be written as:

$$\Rightarrow \overrightarrow{\mathrm{OB}} \cdot \overrightarrow{\mathrm{OC}} = \cos a$$

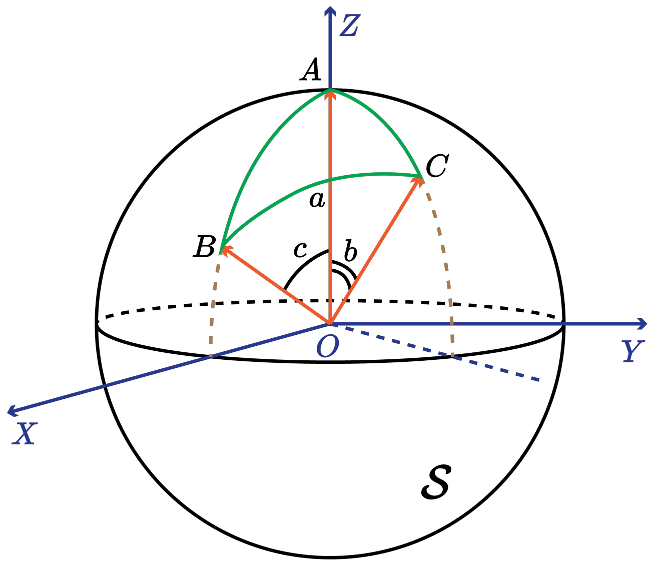

Figure 1.2: The rearranged triangle

$$\Rightarrow \cos a = \cos b \cos c + \sin b \sin c \cos A \quad q.e.d.$$

The proof for ii) will be left as an exercise for the reader. We apply identity i) to the polar triangle and use the relationship between the two triangles. Next, we will prove iii).

For this, we will use the fact that $\sin^2 A = 1 - \cos^2 A$. The expression for $\cos A$ follows from i) as

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

Substituting this, we get

$$\sin^2 A = \frac{\sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$\Rightarrow \frac{\sin a}{\sin A} = \frac{\sin a \sin b \sin c}{\sqrt{\sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2}},$$

which is invariant to circular permutations of a, b and c. Thus, we get

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \quad q.e.d.$$

Definition 2. Let ΔABC be a spherical triangle on a sphere $\mathcal{S}$, generated by a trihedron $OXYZ$ (Fig. 1.3). The triangle $\Delta A'B'C'$ generated by the trihedron $OX'Y'Z'$ on the same sphere $\mathcal{S}$ is called the *polar triangle* of ΔABC if the following are true:

$$\begin{cases} OX' \perp YOZ \\ OY' \perp XOZ \\ OZ' \perp XOY \end{cases} \tag{1.3}$$

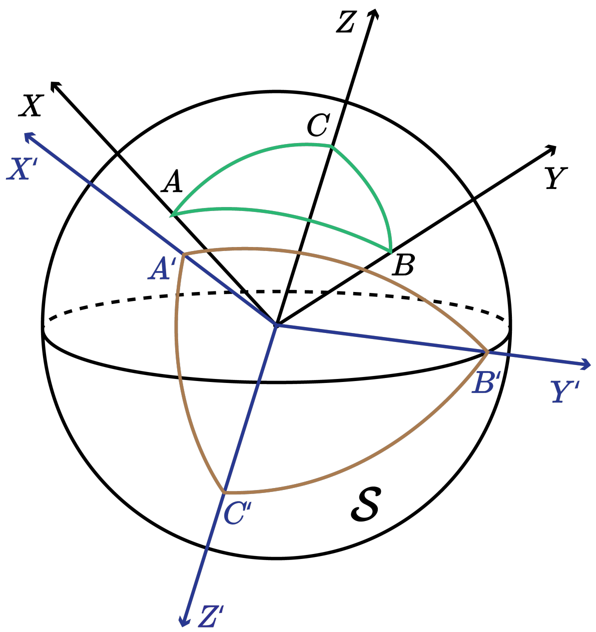

Figure 1.3: Image of ΔABC and its polar triangle $\Delta A'B'C'$

Theorem. Relationship with the Polar Triangle. Let ΔABC be a spherical triangle and $\Delta A'B'C'$ its polar triangle. The following equation applies:

$$A + a' = B + b' = C + c' = \pi \tag{1.4}$$

All of the results above apply to the polar triangle as well.

1.2 Conic Sections

Definition. A *conic section* is a curve obtained from the intersection of the surface of a cone with a plane (Fig. 1.4). Depending on the angle of the plane, there are three types of conic sections: ellipse, parabola, hyperbola. The circle is a special case of the ellipse, though it will be treated separately here due to its orbital properties.

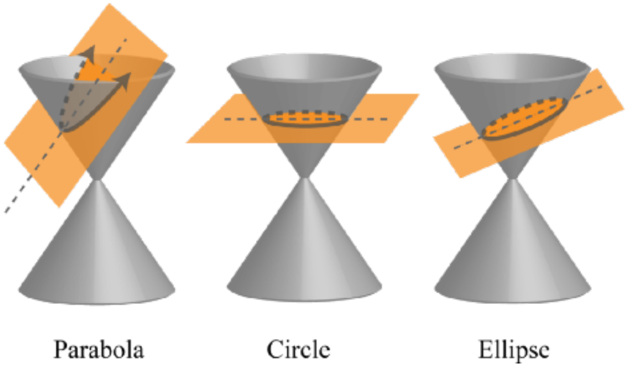

Figure 1.4: 3D view of the four conic sections
`https://www.geogebra.org/resource/x4hgF2Fd/HGmh21K7Wb8w3mxc/material-x4hgF2Fd.png`

The general equation of a conic in polar coordinates is

$$r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta} \tag{1.5}$$

The eccentricity e defines the type of curve and will be addressed in each particular case.

1.2.1 Circle

Definition. A *circle* is the locus of all points equidistant from a single point. The point is called the *center*, while the distance is called the *radius*. It is an ellipse with the eccentricity $e = 0$.

If r is the radius of a circle centered in the point $C(a, b)$, the points lying on the circle obey the equation:

$$\sqrt{(x-a)^2 + (y-b)^2} = r \tag{1.6}$$

If the circle is centered in the origin, i.e. $a = b = 0$, the equation becomes

$$\sqrt{x^2 + y^2} = r \tag{1.7}$$

If one were to plot equation (1.5) in a Cartesian plane, it would look like this:

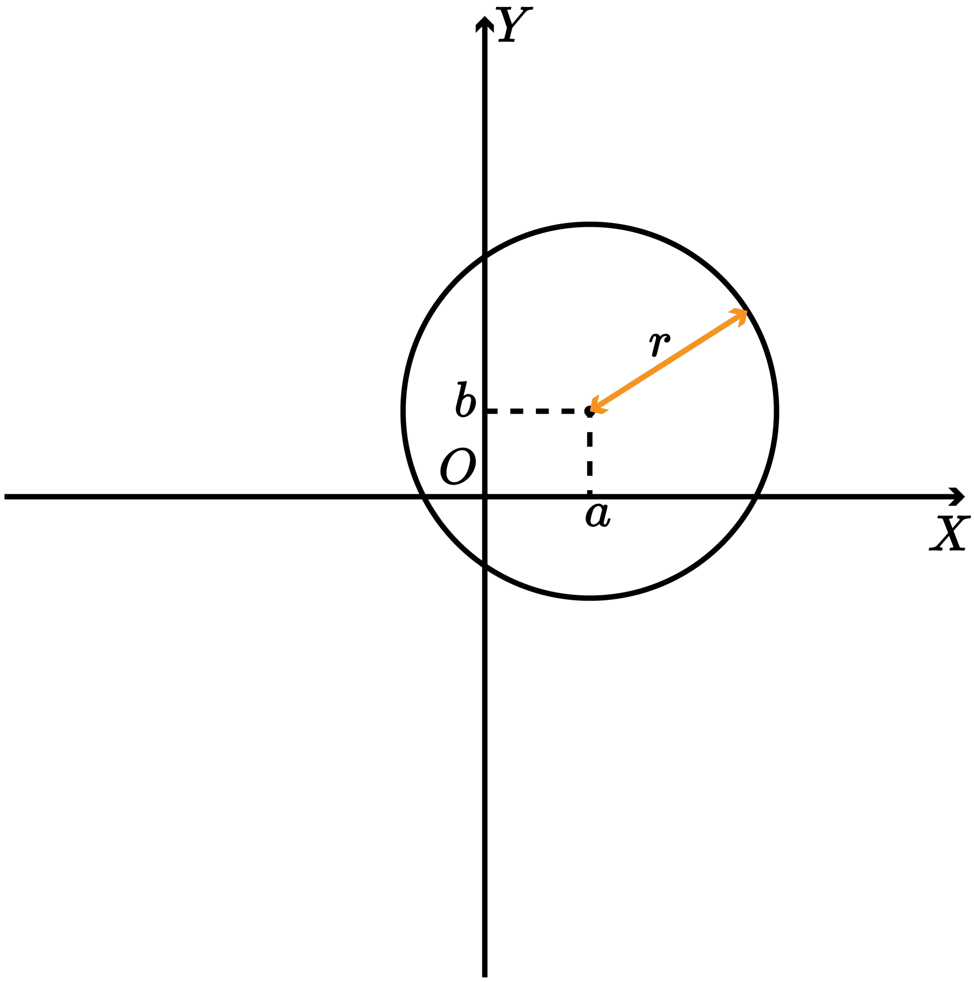

Figure 1.5: Circle Representation

The orbital properties of the circle, together with the other three cases, will be discussed later, when the subject of Orbital Mechanics is put forward. The planets orbit the Sun in elliptical trajectories, but the eccentricities are so small that a circle is a very good approximation. The speed along a circular trajectory is constant.

1.2.2 Ellipse

Definition. An *ellipse* is the locus of all points whose sum of the distances to the two *focal points* is constant. It has the following form in a Cartesian plane:

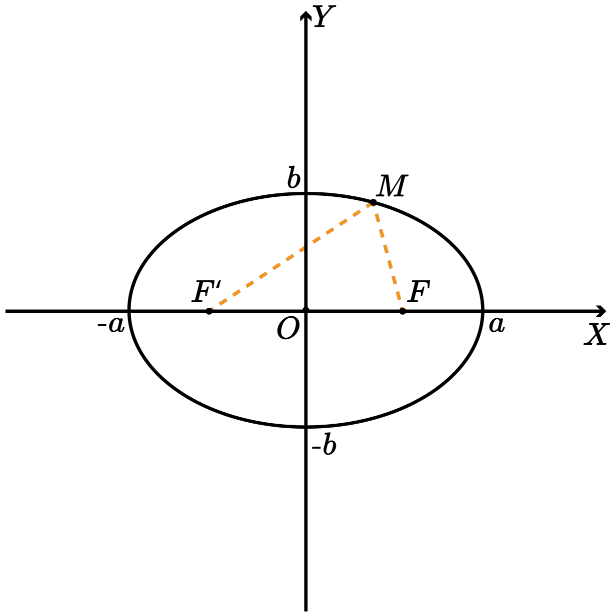

Figure 1.6: Ellipse Representation

The following notations were introduced:

- a - the *semi-major axis*;
- b - the *semi-minor axis*;
- c - the *focal distance*;
- F and F' - the *focal points*;

For an ellipse centered in the point $C(x_0, y_0)$, the equation in Cartesian coordinates is

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1, \tag{1.8}$$

which for $x_0 = y_0 = 0$ becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1.9}$$

The ellipse satisfies the equations:

$$\begin{cases} c = ae \\ a^2 = b^2 + c^2 \Rightarrow b = a\sqrt{1-e^2} \\ \text{M is on the ellipse} \Leftrightarrow MF + MF' = 2a = const. \\ e \in (0,1) \end{cases}$$

1.2.3 Parabola

Definition. A *parabola* is the locus of all points equidistant from a fixed line (called the *directrix*) and a fixed point (called the *focus*). The eccentricity is $e = 1$ and the equation in Cartesian form is

$$y^2 - 2px = 0, \tag{1.10}$$

where p is the *parameter* of the parabola. The representation is

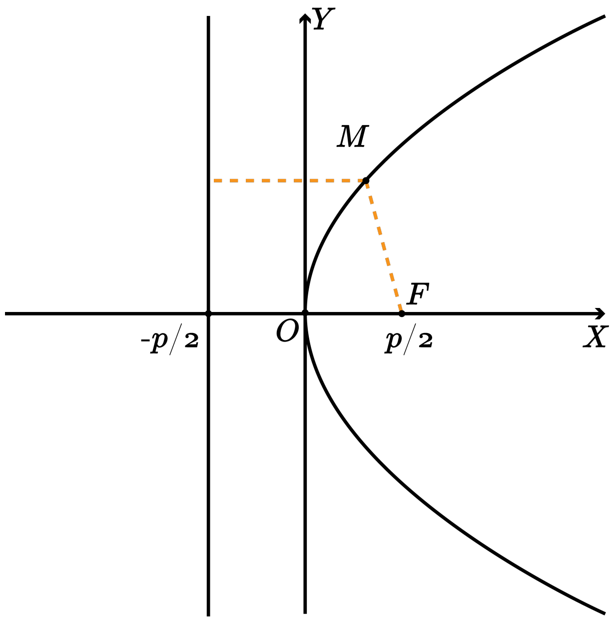

Figure 1.7: Parabola Representation

1.2.4 Hyperbola

Definition 1. A *hyperbola* is the locus of all points whose absolute value of the difference of the distances to the *focal points* is constant. The eccentricity $e > 1$ and the equation in Cartesian coordinates for a hyperbola centered in $C(x_0, y_0)$ is

$$\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} = 1 \tag{1.11}$$

Centering the hyperbola in the origin transforms the equation into

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{1.12}$$

It has the following form in Cartesian coordinates:

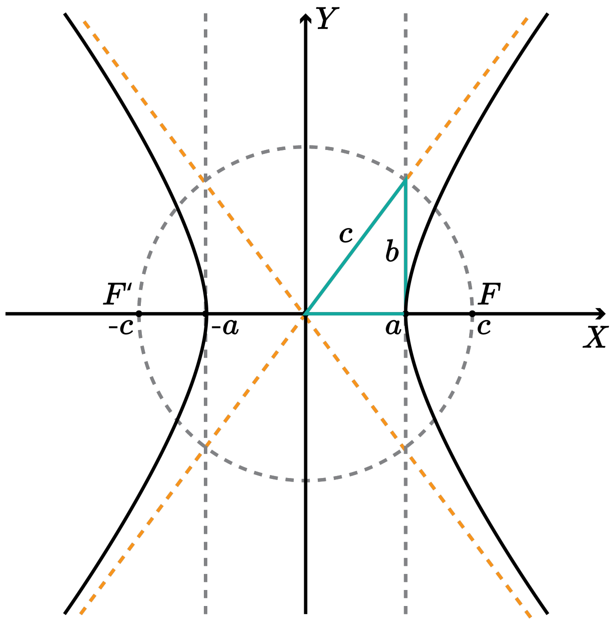

Figure 1.8: Hyperbola Representation

Given the Cartesian representation, it is a trivial proof (and it *will* be left as an exercise to the reader) that the asymptotes are of equation

$$y = \pm\frac{b}{a}x \tag{1.13}$$

Definition 2. Given a hyperbola of equation (1.12), its so-called *conjugate hyperbola* is defined as the hyperbola of equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \tag{1.14}$$

It intersects the Y axis in the points $(0, b)$ and $(0, -b)$ and it can be easily shown that it has the same asymptotes as the so-called *true* hyperbola:

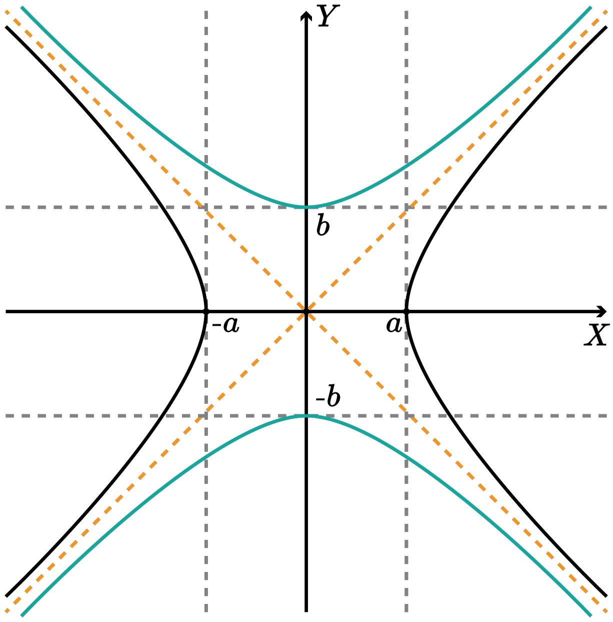

Figure 1.9: *True* Hyperbola in Black, *Conjugate* Hyperbola in Turqoise

For a=b, the hyperbola is called *rectangular* and it obeys the expression:

$$x^2 - y^2 = a^2 \tag{1.15}$$

and has asymptotes

$$y = \pm x \tag{1.16}$$

With this, the part dedicated to conic sections ends. Next, we shall explore the world of Real Analysis, which is an indispensable tool for Physics and Astronomy.

1.3 Real Analysis

This chapter is dedicated to introducing general notions which will be useful not only to Astronomy, but to most branches of Physics. After a short introduction into vector spaces, we shall present the most important results in vector calculus, differential and integral calculus on Banach spaces, while also introducing the apparatus of differential forms in order to bring these results into the modern context.

1.3.1 Sequences and Series of Real Numbers

Definition 1. A sequence of real numbers $(x_n)_n$ is called *convergent to* $x_0 \in \mathbb{R}$ if

$$\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N}^* \text{ such that } \forall n > N_\varepsilon,\ |x_n - x_0| < \varepsilon \tag{1.17}$$

Definition 2. A sequence of real numbers $(x_n)_n$ is called *fundamental* or *Cauchy* if

$$\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N}^* \text{ such that } \forall m, n > N_\varepsilon,\ |x_n - x_m| < \varepsilon \tag{1.18}$$

Theorem. Cauchy's General Convergence Criterion. A sequence of real numbers $(x_n)_n$ is convergent *if and only if* it is fundamental. On this premise, the next result holds true.

Theorem. Every real number is the limit of a (fundamental) sequence of rational numbers, i.e. $\mathbb{Q}$ is dense in $\mathbb{R}$.

This statement will be considered again when the topic of *metric spaces* is put forward, since it affirms that $\mathbb{R}$, together with the Euclidean norm, form a Banach space.

Definition 3. Let $(x_k)_k$ be a sequence of real numbers. The *sequence of partial sums* is defined as follows:

$$S_n = \sum_{k=1}^{n} x_k \tag{1.19}$$

If the limit

$$\lim_{n\to\infty} S_n \overset{\text{not}}{=} S$$

exists and is finite, then we say that the series

$$\sum_{k=1}^{\infty} x_k = \lim_{n\to\infty} \sum_{k=1}^{n} x_k \tag{1.20}$$

is convergent and

$$\sum_{k=1}^{\infty} x_k = S. \tag{1.21}$$

On the contrary, we say that the series in divergent.

Theorem. Cauchy's General Convergence Criterion. A series is convergent *if and only if* the sequence of partial sums is a Cauchy sequence, i.e.

$$\begin{aligned} &\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N} \text{ such that } \forall m, n \in \mathbb{N},\ n > N_\varepsilon, \\ &|x_{n+1} + x_{n+2} + ... + x_{n+m}| < \varepsilon \end{aligned} \tag{1.22}$$

Proposition. If a series is convergent, then the limit of the general term is 0. Equivalently, if the limit of the general term is not 0, then the series is divergent, i.e.

$$\begin{aligned} &\sum_{k=1}^{\infty} x_k < \infty \Rightarrow \lim_{k\to\infty} x_k = 0 \text{ and} \\ &\lim_{k\to\infty} x_k \neq 0 \Rightarrow \sum_{k=1}^{\infty} x_k = \infty. \end{aligned} \tag{1.23}$$

Convergence Criteria. For the following statements we shall assume that

$$\sum_{k\geq 1} a_k \text{ and } \sum_{k\geq 1} b_k \tag{1.24}$$

are two series of positive terms.

First Criterion:

$$\begin{cases} a_k \leq b_k, \forall k \\ \sum_{k\geq 1} b_k < \infty \end{cases} \Rightarrow \sum_{k\geq 1} a_k < \infty \tag{1.25}$$